ASYMPTOTICS OF EIGENVALUES OF THE TWO-PARTICLE SCHRÖDINGER OPERATORS ON LATTICES.

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ABSTRACT. The Hamiltonian of a system of two quantum mechanical particles moving on the d-dimensional lattice $\mathbb{Z}^d$ and interacting via zero-range attractive pair potentials is considered. For the two-particle energy operator $H_\mu(K)$, $K \in \mathbb{T}^d = (-\pi, \pi]^d$ – the two-particle quasi-momentum, the existence of a unique positive eigenvalue $z(\mu, K)$ above the upper edge of the essential spectrum of $H_\mu(K)$ is proven and asymptotics for $z(\mu, K)$ are found when μ approaches to some $\mu_0(K)$ and $K \to 0$.

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1. INTRODUCTION

In this paper we will consider the family of the two-particle Schrödinger operators associated to a system of two identical particles moving on d - dimensional cubic lattice $\mathbb{Z}^d$ and interacting via zero-range potentials. In the momentum representation the corresponding operator is of the form

$$H_\mu(K) = H_0(K) + \mu V, \quad K \in \mathbb{T}^d,$$

where $\mathbb{T}^d$ – d dimensional torus. The non perturbed operator $H_0(K)$ is the multiplication operator by the function

$$\mathcal{E}_K(q) = \varepsilon\left(\frac{K}{2} + q\right) + \varepsilon\left(\frac{K}{2} - q\right),$$

where

$$\varepsilon(q) = \sum_{i=1}^{d}(1 - \cos q_i), \tag{1.1}$$

V is integral operator of rank one and $\mu > 0$ is repulsive interaction. $H_\mu(K)$ has continuous spectrum $[\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)]$ and at most one eigenvalue $z(\mu, K)$ on the right from $\mathcal{E}_{\max}(K)$.

In celebrated work [1] of B.Simon and M.Klaus it is considered a family of Schrödinger operators $H = -\Delta + \lambda V$ and, a situation where as $\lambda \downarrow \lambda_0$ some eigenvalue $e_i(\lambda) \uparrow 0$, i.e., as $\lambda \downarrow \lambda_0$ an eigenvalue is absorbed into continuous

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spectrum, and conversely, as $\lambda \uparrow \lambda_0 + \varepsilon$ continuous spectrum "gives birth" to a new eigenvalue. This phenomenon in [1] is called "coupling constant threshold".

The phenomenon coupling constant threshold is a significant tool not only for the two-particle continuous Hamiltonians (see [1]), but also in the existence of the three-particle bound states of the Hamiltonians of a system of three particles, in particular, for the Efimov effect (see [2, 3, 4]).

In the case of the three-particle lattice Schrödinger operators $\boldsymbol{H}_{\mu_0}(K),\, K \in \mathbb{T}^3$ – three-particle quasi-momentum, associated to the Hamiltonian of a system of three particles on $\mathbb{Z}^3$ interacting via zero-range pair potentials $\mu < 0$ the following phenomenon is also deeply related to the coupling constant threshold $\mu_0 < 0$ of the two-particle Schrödinger operators: for $\mu = \mu_0$ the corresponding three-particle lattice Schrödinger operator $\boldsymbol{H}_{\mu_0}(0)$ has infinitely many eigenvalues, whereas $\boldsymbol{H}_{\mu_0}(K),\, K \neq 0$ has only finitely many (see [5, 6]).

Throughout physics, stable composite objects are usually formed by way of attractive forces, which allow the constituents to lower their energy by binding together. Repulsive forces separate particles in free space. However, in structured environment such as a periodic potential and in the absence of dissipation, stable composite objects can exist even for repulsive interactions (see [10]).

In the present paper, for the two-particle operator $H_\mu(K),\, K \in \mathbb{T}^d$, it is established: at first, if $d \geq 3$ then there exists such $\mu_0 = \mu_0(K) > 0$ (coupling constant threshold) that the operator has non eigenvalue for any $0 < \mu < \mu_0$, but for any $\mu > \mu_0$ there is a unique eigenvalue $z(\mu, K)$ of $H_\mu(K)$ lying above the upper edge of $\sigma_{\mathrm{ess}}(H_\mu(K))$. In [1], it is only assumed the existence of the coupling constant threshold $\lambda_0 > 0$, but in this work, the coupling constant threshold is definitely found by the given data.

Secondly, as in [1], two questions will concern us: (i) For fixed $K \in \mathbb{T}^d$, is $z(\mu, K)$ analytic at $\mu = \mu_0$, if singular, does it have an expansion in some singular quality like $(\mu - \mu_0)^\alpha,\, \alpha \in \mathbb{R}$? (ii) For fixed $K \in \mathbb{T}^d$, what is the rate at which $z(\mu, K)$ approaches to the upper edge of the essential spectrum $\mathcal{E}_{\max}(K)$, as μ approaches to $\mu_0(K)$?

Thirdly, lattice Schrödinger operators are strictly dependant on the quasi-momentum K of the system, and it is one of the differences between continuous and lattice operators. Thus, we are interested in one more question: (iii) What is the rate at which $z(\mu_0(0), K)$ approaches to $\mathcal{E}_{\max}(0)$ as $K \to 0$?

The paper is organized as follows.

In Section 2 we introduce the two-particle operator $H_\mu(K)$ and give a location of its essential spectrum.

In Section 3 we define coupling constant threshold $\mu_0(K)$ in some concrete domain $\Pi_0 \subset \mathbb{T}^d$ and give main results of the paper.

In Section 4 we prove main results.

In Appendix for reader's convenience, we give full proves of some consequences of the implicit function theorem, used in the proof of Theorem 3.2 and proves of some lemmas, used in the proof of Lemma 4.2.

2. The two-particle operator $H_\mu(K)$ and its Essential spectrum

Let $\mathbb{Z}^d$ be d dimensional hybercubic lattice and $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d = (-\pi, \pi]^d$ be d dimensional torus (Brillouin zone), the dual group of $\mathbb{Z}^d$. Denote by $L_2(\mathbb{T}^d, d\eta)$ the Hilbert space of square-integrable functions on $\mathbb{T}^d$, where $d\eta$ – Haar measure in $\mathbb{T}^d$.

Let $L_2^e(\mathbb{T}^d)$ the subspace of even functions in $L_2(\mathbb{T}^d, d\eta)$. Consider the analytic function on $\mathbb{T}^d$

$$\varepsilon(q) = \sum_{i=1}^{d} (1 - \cos q_i).$$

In the momentum representation the two-particle Hamiltonians are given by the bounded self-adjoint operators on the Hilbert space $L_2^e(\mathbb{T}^d)$ as following (see [7]):

$$H_\mu(K) = H_0(K) + \mu V. \tag{2.1}$$

The non-perturbed operator $H_0(K)$ on $L_2^e(\mathbb{T}^d)$ is multiplication operator by the function $\mathcal{E}_K(\cdot)$:

$$(H_0(K)f)(q) = \mathcal{E}_K(q) f(q), \quad f \in L_2^e(\mathbb{T}^d), \tag{2.2}$$

where

$$\mathcal{E}_K(q) = \frac{1}{m}\left[\varepsilon\left(\frac{K}{2} - q\right) + \varepsilon\left(\frac{K}{2} + q\right)\right] = \frac{1}{m}\sum_{j=1}^{d}\left[2 - 2\cos\left(\frac{K_j}{2}\right)\cos q_j\right], \tag{2.3}$$

The perturbation V is an integral operator of rank one

$$(Vf)(p) = \int_{\mathbb{T}^d} f(q) d\eta, \quad f \in L_2(\mathbb{T}^d).$$

Further without loss of generality we assume that $m = 1$.

The perturbation V of the multiplication operator $H_0(K)$ is a self-adjoint operator of rank one. Therefore in accordance to Weil's theorem (see [9]), the essential spectrum of $H_\mu(K)$, $K \in \mathbb{T}^d$ fills the following interval on the real axis:

$$\sigma_{ess}(H_\mu(K)) = \sigma(H_0(K)) = [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)],$$

where

$$\mathcal{E}_{\min}(K) \equiv \min_p \mathcal{E}_K(p) = \sum_{j=1}^{d}\left[2 - 2\cos\left(\frac{K_j}{2}\right)\right],$$

$$\mathcal{E}_{\max}(K) \equiv \max_p \mathcal{E}_K(p) = \sum_{j=1}^{d}\left[2 + 2\cos\left(\frac{K_j}{2}\right)\right].$$

Remark 2.1. *We remark that the essential spectrum $[\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)]$ strongly depends on the quasi-momentum $K \in \mathbb{T}^d$; when $K = \vec{\pi} = (\pi, \pi, ..., \pi) \in \mathbb{T}^d$ the essential spectrum of $H_\mu(K)$ degenerated to the set consisting of a unique point*

$\{\mathcal{E}_{\min}(\vec{\pi}) = \mathcal{E}_{\max}(\vec{\pi}) = 2d\}$ *and hence the essential spectrum of* $H_\mu(K)$ *is not absolutely continuous for all* $K \in \mathbb{T}^d$.

3. Main Results

Set

$$\Pi_n = \Big\{ k \in \mathbb{T}^d : \text{ at most } n \text{ coordinates of } k \text{ is equal to } \pi \Big\}, \quad 0 \le n \le d.$$

It is clear that $\Pi_d = \mathbb{T}^d$, $\Pi_m \subset \Pi_n$ if $m < n$ and

$$\Pi_0 = \Big\{ k = (k_1, \dots, k_d \in \mathbb{T}^d : \, k_i \neq \pi, \; i = 1, \dots, d) \Big\}.$$

Let $\mathbb{C}$ be the complex plane. For any $K \in \mathbb{T}^d$ we define a regular function $\nu(K, \cdot)$ in $\mathbb{C} \backslash [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)]$ by

$$\nu(K, z) = \int\limits_{\mathbb{T}^d} \frac{d\eta}{z - \mathcal{E}_K(q)}. \tag{3.1}$$

For $d \ge 3$ and $K \in \Pi_{d-3}$, the function $\mathcal{E}_K(q)$ has a unique non degenerated maximum and, consequently, the following integral exists and defines analytic function on Π_0 :

$$\nu(K) = \nu(K, \mathcal{E}_{\max}(K)) = \int\limits_{\mathbb{T}^d} \frac{d\eta}{\mathcal{E}_{\max}(K) - \mathcal{E}_K(q)} \tag{3.2}$$

(see Lemma 4.2).

Remark 3.1. *If* n *coordinates of* K *equals to* π *then* $\mathcal{E}_{\max}(K) - \mathcal{E}_K(q)$ *in* (3.2) *can be considered as the function defined on* $\mathbb{T}^{d-n}$ *having non-degenerated maximum at* $\vec{\pi} = (\pi, \dots, \pi) \in \mathbb{T}^{d-n}$. *Therefore without loss of generality we can always assume that any coordinates of* $K \in \mathbb{T}^d$ *is not equal to* π, *that is,* $K \in \Pi_0 \subset \mathbb{T}^d$.

Let $d \ge 3$. Define the positive number $\mu_0(K)$, $K \in \Pi_0$ as

$$\mu_0(K) = \frac{1}{\nu(K)} = \left(\int\limits_{\mathbb{T}^d} \frac{d\eta}{\mathcal{E}_{\max}(K) - \mathcal{E}_K(q)} \right)^{-1}. \tag{3.3}$$

The following theorem is on the existence and dependance of eigenvalues of the two-particle operator $H_\mu(K)$ on interaction energy $\mu > 0$. In fact, we prove that there exists a unique eigenvalue $z(\mu, K)$ of $H_\mu(K)$, $K \in \Pi_0$ above its essential spectrum depending on the coupling constant $\mu_0 = \mu_0(K) > 0$. Moreover we find an expansion for the difference

$$z(\mu, K) - \mathcal{E}_{\max}(K)$$

at $\mu = \mu_0$. This expansion is highly dependant on dimension $d \ge 3$ of the quasi-momentum $K \in \mathbb{T}^d$: $1)$ if $d = 3$, then $z(\mu, K) - \mathcal{E}_{\max}(K)$ is an analytic function of $\mu - \mu_0$ with the leading term of $(\mu - \mu_0)^2$; $2)$ if $d = 4$, then the difference does not expand to Puizo series, but it has an expansion with the first term of

$\sigma = (\mu - \mu_0)(-\ln(\mu - \mu_0))^{-1}$; 3) if $d \geq 5$ and odd, then $z(\mu, K) - \mathcal{E}_{\max}(K)$ is and analytic function of $\alpha = (\mu - \mu_0)^{1/2}$ and the leading term of the expansion is $\mu - \mu_0$; 4) if $d \geq 6$ and even, then the expansion of $z(\mu, K) - \mathcal{E}_{\max}(K)$ is an analytic function of

$$\sigma = (\mu - \mu_0)^{1/2}, \quad \tau = (\mu - \mu_0)^{1/2} \ln(\mu - \mu_0)^{1/2}$$

with the first term of $\mu - \mu_0$.

Theorem 3.2. *Let $d \geq 3$. Then for any $K \in \Pi_{d-3}$ and $\mu > \mu_0(K)$ the operator $H_\mu(K)$ has a unique eigenvalue $z(\mu, K)$ lying above the upper edge $\mathcal{E}_{\max}(K)$ of the essential spectrum of $H_\mu(K)$. Moreover, for any $K \in \Pi_0$ the relation $\mu \to \mu_0(K)$ holds if and only if $z(\mu, K) \to z(\mu_0(K), K) = \mathcal{E}_{\max}(K)$ and for $\mu - \mu_0(K)$ sufficiently small and positive, the difference $z(\mu, K) - z(\mu_0(K), K)$ has following absolutely convergent expansions:*

(i) *if $d = 3$, then*

$$z(\mu, K) - z(\mu_0(K), K) = \left(\sum_{n=1}^{\infty} c_n(K)[\mu - \mu_0(K)]^n \right)^2,$$

where $c_n(K)$, $n = 1, 2, \ldots$ is a real number with

$$c_1(K) = \left(\frac{\pi(\mu_0(K))^2 \Phi_0(K)}{2} \right)^{-1}$$

and

$$\Phi_0(K) = \frac{c}{\sqrt{\cos \frac{K_1}{2} \ldots \cos \frac{K_d}{2}}}, \quad c = const;$$

(ii) *if $d = 4$ then*

$$z(\mu, K) - z(\mu_0(K), K) = \sum_{n \geq 1,\, m,k,l \geq 0} c(n, m, k, l; K)\, \sigma^n\, \tau^m\, \omega^k\, \lambda^l \tag{3.4}$$

with

$$\sigma = \frac{\mu - \mu_0(K)}{-\ln(\mu - \mu_0(K))}, \quad \tau = \frac{1}{-\ln(\mu - \mu_0(K))}, \quad \omega = \frac{\ln \ln(\mu - \mu_0(K)^{-1}}{-\ln(\mu - \mu_0(K))}, \quad \lambda = \mu - \mu_0(K)$$

and $c(n, m, k, l; K)$, $n = 1, 2, \ldots$, $m, k, l = 0, 1, 2 \ldots$ is a real number with

$$c(1, 0, 0, 0; K) = \frac{2}{(\mu_0(K))^2 \Phi_0(K)};$$

(iii) *if $d \geq 5$ and odd, then*

$$z(\mu, K) - z(\mu_0(K), K) = \left(\sum_{n \geq 1} c_n(K)(\mu - \mu_0(K))^{n/2} \right)^2,$$

where $c_n(K)$, $n = 1, 2, \ldots$ is a real numbers with

$$c_1(K) = \left(-(\mu_0(K))^2\, \frac{\partial \nu}{\partial z}(K, \mathcal{E}_{\max}(K)) \right)^{-1/2}$$

and $\nu(\cdot,\cdot)$ is defined by (3.1)*;*

(iv) *if $d \geq 6$ and even, then*

$$z(\mu, K) - z(\mu_0(K), K) = \left(\sum_{n\geq 1,\, m\geq 0} c(n,m;K)\, \sigma^n\, \tau^m \right)^2 \tag{3.5}$$

with

$$\sigma = (\mu - \mu_0(K))^{1/2}, \quad \tau = (\mu - \mu_0(K))^{1/2} \ln(\mu - \mu_0(K))^{1/2}$$

where $c(n,m;K)$, $n = 1,2,\dots$, $m = 0,1,2,\dots$ is a real number with

$$c(1,0;K) = \left(-(\mu_0(K))^2 \, \frac{\partial \nu}{\partial z}(K, \mathcal{E}_{\max}(K)) \right)^{-1/2}.$$

In the following theorem, we show existence and describe dependance of eigenvalues of $H_{\mu^0}(K)$, $\mu^0 = \mu_0(0)$ on the quasi-momentum K : for any $K \in \Pi_0 \setminus \{0\}$ there exists a unique eigenvalue $z(\mu^0, K)$ of the operator and we find an asymptotics of the difference $z(\mu^0, K) - \mathcal{E}_{\max}(0)$ as $K \to 0$.

Theorem 3.3. *For any $K \in \Pi_0 \setminus \{0\}$, the operator $H_{\mu_0(0)}(K)$ has a unique eigenvalue $z(\mu_0(0), K)$ lying above the essential spectrum $\sigma_{ess}(H_{\mu_0(0)}(K))$. Moreover for $z(\mu_0(0), K)$ the following asymptotics hold:*

(i) *if $d = 3$, then,*

$$z(\mu_0(0), K) - \mathcal{E}_{\max}(0) = -\frac{1}{4}|K|^2 + O(|K|^4), \ K \to 0;$$

(ii) *if $d = 4$, then*

$$z(\mu_0(0), K) - \mathcal{E}_{\max}(0) = -\frac{1}{4}|K|^2 + o(|K|^2), \ K \to 0;$$

(iii) *if $d \geq 5$, then*

$$z(\mu_0(0), K) - \mathcal{E}_{\max}(0) = \alpha |K|^2 + o(|K|^2), \ K \to 0,$$

where

$$\alpha = -\left(\frac{\partial^2 \nu(0)}{\partial K_1^2} \right) \left(\frac{\partial \nu}{\partial z}(0, \mathcal{E}_{\max}(0)) \right)^{-1} - \frac{1}{4},$$

and $\nu(\cdot)$ is defined by (3.2).

4. The proof of the results

For any $K \in \mathbb{T}^d$, we define the Fredholm determinant associated to the operator $H_\mu(K)$ as analytic function in $z \in \mathbb{C} \setminus [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)]$ by

$$\Delta_\mu(K, z) = 1 - \mu\nu(K, z).$$

Observe that the function $\Delta_\mu(K, z)$ is real-analytic in $\mathbb{T}^d \times (\mathbb{R} \setminus [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)])$.

Lemma 4.1. *For any $K \in \mathbb{T}^d$ the eigenvalues of $H_\mu(K)$ outside the essential spectrum coincides with the zeros of $\Delta_\mu(K, \cdot)$.*

Proof. Let $z \in \mathbb{C} \setminus \sigma_{ess}(H_\mu(K))$ be an eigenvalue of $H_\mu(K)$ and $f \in L_2^e(\mathbb{T}^d)$ be one of the eigenvectors corresponding to z. Then by the definition of eigenvalue one can get that

$$(z - \mathcal{E}_K(p))f(p) = \mu \int\limits_{\mathbb{T}^d} f(q)d\eta.$$

Therefore, the equation

$$\phi(p) = \mu \int\limits_{\mathbb{T}^d} \frac{\phi(q)d\eta}{z - \mathcal{E}_K(q)}$$

has a simple solution $\phi(q) \equiv 1$ (up to constant factor) in $L_2^e(\mathbb{T}^d)$. So we obtain that

$$1 = \mu \int\limits_{\mathbb{T}^d} \frac{d\eta}{z - \mathcal{E}_K(q)}$$

and hence $\Delta_\mu(K, z) = 0$.

Conversely, let $z \in \mathbb{C} \setminus \sigma_{ess}(H_\mu(K))$ be a solution of the equation $\Delta_\mu(K, z) = 0$. Then it is easy to see that the function $f(\cdot) = (z - \mathcal{E}_K(\cdot))^{-1} \in L_2^e(\mathbb{T}^d)$ satisfies the equality $H_\mu(k)f = z\,f$, which means z is eigenvalue (see also [5]). $\square$

The main results of the paper and their proves are based on the following Lemma (see [8]), which is important tool not only in studying spectral properties of the two-particle Schrödinger operators, but also in the spectral analysis of the three-particle Schrödinger operators (see [5], [6]).

Lemma 4.2. (i) *If $d \geq 3$ and $K \in \Pi_0$, the integral*

$$\nu(K) = \nu(K, \mathcal{E}_{\max}(K)) = \int\limits_{\mathbb{T}^d} \frac{d\eta}{\mathcal{E}_{\max}(K) - \mathcal{E}_K(q)} \tag{4.1}$$

exists and defines analytic function on $\Pi_0 \subset \mathbb{T}^d$.

(ii) *Let $K \in \Pi_0$ and $z > \mathcal{E}_{\max}(K)$. The function $\nu(K, z)$ can be represented as*

$$\begin{aligned}
\nu(K,z) &= -\frac{\Phi_0(K)}{2}\,(\mathcal{E}_{\max}(K) - z)^m \ln(z - \mathcal{E}_{\max}(K)) + \\
&+ (\mathcal{E}_{\max}(K) - z)^{m+1} \ln(z - \mathcal{E}_{\max}(K))\Phi_{11}(K,z) + \nu(K) + \Phi_2(K,z),
\end{aligned}$$

if $d = 2m + 2$ and

$$\begin{aligned}
\nu(K,z) &= \frac{\pi\Phi_0(K)}{2}\,\frac{(\mathcal{E}_{\max}(K) - z)^m}{\sqrt{z - \mathcal{E}_{\max}(K)}} + \\
&+ (\mathcal{E}_{\max}(K) - z)^{m+1/2}\,\Phi_{12}(K,z) + \nu(K) + \Phi_2(K,z),
\end{aligned}$$

if $d = 2m + 1$, where

$$\Phi_0(K) = \frac{c}{\sqrt{\cos\dfrac{K_1}{2} \ldots \cos\dfrac{K_d}{2}}}, \quad c = const,$$

and

$$\Phi_{12}(K,z)=\sum_{l=0}^{\infty} b_l(K)\,(z-\mathcal{E}_{\max}(K))^{l/2}$$

and $\Phi_{11}(K,\cdot)$ *and* $\Phi_2(K,\cdot)$, $K\in\Pi_0$ *are analytic functions in some neighborhood* $V_\varepsilon(\mathcal{E}_{\max}(K))$ *of* $z=\mathcal{E}_{\max}(K)$, $\Phi_2(K,\mathcal{E}_{\max}(K))=0$ *and* $b_l(K)$, $l=0,1,2,\dots$ *are some real coefficients.*

Proof of Lemma 4.2. (i) It is easy to see that $q_0=\vec{\pi}$ is a unique non degenerate maximum of $\mathcal{E}_K(q)$. We rewrite function $\nu(K,z)$ as

$$\nu(K,z)=\int\limits_{U_\delta(\vec{\pi})}\frac{d\eta}{z-\mathcal{E}_K(q)}+\int\limits_{\mathbb{T}^d\setminus U_\delta(\vec{\pi})}\frac{d\eta}{z-\mathcal{E}_K(q)}= \\ =G_1(K,z)+G_2(K,z). \tag{4.2}$$

Observe that $G_2(\cdot;z)$, $z>\mathcal{E}_{\max}(K)$ and $G_2(K,\cdot)$, $K\in\Pi_0$ are analytic functions on Π_0 and $(\mathcal{E}_{\max}(K),\infty)$ respectively. Moreover observe that $G_2(K,\mathcal{E}_{\max}(K))$ is regular on Π_0.

In the first integral making a change of variables $q=\phi(x)$, where

$$\phi: U_\delta(\vec{\pi})\to W_\gamma(0),\quad q_j:\to \arccos\left(\frac{x_j^2}{2\cos\frac{K}{2}}-1\right),$$

we get:

$$G_1(K,z)=\int\limits_{W_\gamma(0)}\frac{1}{z-\mathcal{E}_{\max}(K)+\sum\limits_{j=1}^{d}x_j^2}\times$$

$$\times\frac{2dx_1}{\sqrt{2\cos\dfrac{K_1}{2}}\sqrt{2-\dfrac{x_1^2}{2\cos\dfrac{K_1}{2}}}}\cdots\frac{2dx_d}{\sqrt{2\cos\dfrac{K_d}{2}}\sqrt{2-\dfrac{x_d^2}{2\cos\dfrac{K_d}{2}}}}.$$

There is no loss of generality in assuming that $W_\gamma(0)$ is sphere in $\mathbb{R}^d$ with center at $x=0$ and with radius $\gamma>0$.

Passing spherical coordinates as

$$\begin{aligned} x_1 &= r\cos\psi_1\cos\psi_2\dots\cos\psi_{d-2}\cos\psi_{d-1}\\ x_2 &= r\cos\psi_1\cos\psi_2\dots\cos\psi_{d-2}\sin\psi_{d-1}\\ x_3 &= r\cos\psi_1\cos\psi_2\dots\sin\psi_{d-2}\\ &\vdots\\ x_d &= r\sin\psi_1 \end{aligned}$$

$$0\le r\le\gamma,\quad 0\le\psi_1\le 2\pi,\quad -\frac{\pi}{2}\le\psi_2\le\frac{\pi}{2},\ \dots,\ -\frac{\pi}{2}\le\psi_d\le\frac{\pi}{2}$$

we obtain

$$G_1(K,z)=\frac{1}{\sqrt{\cos\frac{K_1}{2}\ldots\cos\frac{K_d}{2}}}\times \tag{4.3}$$

$$\times\int\limits_0^{\gamma}\int\limits_{-\pi/2}^{\pi/2}\int\limits_{-\pi/2}^{\pi/2}\cdots\int\limits_0^{2\pi}\frac{r^{d-1}\omega(\psi)}{z-\mathcal{E}_{\max}(K)+r^2}\frac{dr\,d\psi_{d-1}\ldots d\psi_1}{\sqrt{1-\dfrac{r^2\omega_1^2(\psi)}{2\cos\dfrac{K_1}{2}}}\cdots\sqrt{1-\dfrac{r^2\omega_d^2(\psi)}{2\cos\dfrac{K_d}{2}}}},$$

where

$$\omega(\psi)=\omega(\psi_1,\ldots,\psi_{d-1})=\cos^{d-2}\psi_{d-1}\ldots\cos\psi_2,$$
$$\omega_1(\psi)=\omega(\psi_1,\ldots,\psi_{d-1})=\cos\psi_1\cos\psi_2\ldots\cos\psi_{d-1}$$

and

$$\omega_j(\psi)=\omega(\psi_1,\ldots,\psi_{d-1})=\cos\psi_1\cos\psi_2\ldots\sin\psi_{d-j+1},\ j=2,\ldots,d.$$

Since

$$\frac{1}{\sqrt{1-\dfrac{x^2}{2}}}=1+\frac{1}{4}x^2+\frac{3}{32}x^4+\ldots,\quad \text{if } |x|<\sqrt{2},$$

it is easy to see that

$$\frac{1}{\sqrt{1-\dfrac{r^2\omega_j^2(\psi)}{2\cos\frac{K_j}{2}}}}=1+\frac{1}{4}\frac{\omega_j(\psi)}{\sqrt{\cos\frac{K_j}{2}}}r^2+\frac{3}{32}\frac{\omega_j^2(\psi)}{\cos\frac{K_j}{2}}r^4+\ldots,\quad j=1,\ldots,d. \tag{4.4}$$

Observe that these series converges uniformly in any compact set $\mathbf{K}\subset\Pi_0$ for any $r\in[0,\gamma]$. Therefore the series obtained by term by term multiplication of these series

$$\frac{1}{\sqrt{1-\dfrac{r^2\omega_1^2(\psi)}{2\cos\frac{K_1}{2}}}}\cdots\frac{1}{\sqrt{1-\dfrac{r^2\omega_d^2(\psi)}{2\cos\frac{K_d}{2}}}}=1+\tilde{J}_2(\psi;K)r^2+\tilde{J}_4(\psi;K)r^4+\ldots$$

also converges uniformly in the set $\mathbf{K}\times[0,\gamma]$.

Set

$$C_s(K)=\int\limits_{-\pi/2}^{\pi/2}\int\limits_{-\pi/2}^{\pi/2}\cdots\int\limits_0^{2\pi}\omega(\psi)\tilde{J}_s(\psi;K)d\psi_1\ldots d\psi_{d-1}, \tag{4.5}$$

$s=0,2,4,\ldots$. Note that these functions are analytic functions of K on Π_0 since they depend on only

$$\left(\cos\frac{K_1}{2}\right)^{-1/2},\ldots,\left(\cos\frac{K_d}{2}\right)^{-1/2}.$$

It is easy to see that the series

$$\sum_{s=0}^{\infty} C_{2s}(K) r^{2s}$$

is obtained by integrating uniformly convergent series and so converges uniformly on $\mathbf{K} \times [0, \gamma]$.

Therefore using the expansion of $\cos \frac{K}{2}$ for sufficiently small neighborhood of $K = 0$ one can find the following expansion:

$$\frac{C_s(K)}{\sqrt{\cos \dfrac{K_1}{2} \ldots \cos \dfrac{K_d}{2}}} = c_{s,0} + \sum_{l=1}^{d} c_{s,1,l} K_l^2 + \sum_{l,m=1}^{d} c_{s,2,l,m} K_l^2 K_m^2 + \ldots, \tag{4.6}$$

where $c_{s,\ldots}$ are some real numbers. From this expansion we establish that

$$\Phi_0(K) = \frac{c}{\sqrt{\cos \dfrac{K_1}{2} \ldots \cos \dfrac{K_d}{2}}} = c_{0,0} + O(|K|^2), \text{ as } K \to 0,$$

where $c_{0,0} \neq 0$.

So the function $G_1(K, z)$ can be represented as

$$G_1(K, z) = \frac{1}{\sqrt{\cos \dfrac{K_1}{2} \ldots \cos \dfrac{K_d}{2}}} \Bigg(C_0(K) \int_0^{\gamma} \frac{r^{d-1} dr}{z - \mathcal{E}_{\max}(K) + r^2} + \tag{4.7}$$

$$C_2(K) \int_0^{\gamma} \frac{r^{d+1} dr}{z - \mathcal{E}_{\max}(K) + r^2} + C_4(K) \int_0^{\gamma} \frac{r^{d+3} dr}{z - \mathcal{E}_{\max}(K) + r^2} + \ldots \Bigg).$$

As above said, if $d \geq 3$ for any $K_0 \in \Pi_0$ in the sufficiently small compact neighborhood of $K = K_0$ and for all r small the series

$$\sum_{s=0}^{\infty} C_{2s}(K)\, r^{2s+d-3}$$

converges uniformly. Therefore there exists $\lim\limits_{z \to \mathcal{E}_{\max}(K)} \nu(K, z)$ and

$$\nu(K) = \nu(K, \mathcal{E}_{\max}(K)) = \lim_{z \to \mathcal{E}_{\max}(K)} \nu(K, z) = \tag{4.8}$$

$$= \frac{1}{\sqrt{\cos \dfrac{K_1}{2} \ldots \cos \dfrac{K_d}{2}}} \sum_{s=0}^{\infty} C_s(K) \int_0^{\gamma} r^{2s+d-3}\, dr + \int_{\mathbb{T}^d \setminus U_\delta(\vec{\pi})} \frac{dq}{\mathcal{E}_{\max}(K) - \mathcal{E}_K(q)}.$$

Since latter functions are analytic at any $K \in \Pi_0$ it follows that $\nu(K)$ is analytic on Π_0. ▲

Note that for $K \in U_\delta(0)$, using symmetricalness of $\nu(\cdot)$ we obtain that

$$\nu(K) = \nu(0) + a_2 \sum_{l=1}^{d} K_l^2 + \sum_{l,m=1}^{d} a_{4,l,m} K_l^2 K_m^2 + \dots,$$

where $a_{4,s,m}$ are real numbers, and $a_2 = \dfrac{\partial^2 \nu(0)}{\partial K_1^2}$.

Part (i) of Lemma 4.2 is proved.

(ii) According to (4.3), (4.4) and (4.5), the coefficient $C_0(K)$ does not depend on K. Moreover if $d = 1$ then $C_0(K) = 2$. If $d > 1$ then using relation (B.2) (see Appendix B) we get

$$C_0(K) = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \dots \int_{0}^{2\pi} \cos^{d-2} \psi_{d-1} \dots \cos \psi_2 \, d\psi_1 \dots d\psi_{d-1} =$$

$$= 2\pi \int_{-\pi/2}^{\pi/2} \cos^{d-2} \psi_{d-1} \, d\psi_{d-1} \dots \int_{-\pi/2}^{\pi/2} \cos \psi_2 \, d\psi_2 = 2\pi \, a_{d-2} \dots a_1.$$

Therefore it can be represented as

$$C_0(K) = \begin{cases} 2, & \text{if} \quad d = 1 \\ \dfrac{(2\pi)^{m+1}}{(2m)!!}, & \text{if} \ \ d = 2m+2 \\ \dfrac{(2\pi)^{m+1}}{(2m-1)!!}, & \text{if} \ \ d = 2m+1 \end{cases}$$

Using Lemma B.1 (Appendix B) and relations (4.7) and (4.2), we get the expansion for the function $\nu(K, \cdot)$ at the point $z = \mathcal{E}_{\max}(K)$

$$\begin{aligned} \nu(K,z) \;\; &= \;\; -\frac{\Phi_0(K)}{2} (\mathcal{E}_{\max}(K) - z)^m \ln(z - \mathcal{E}_{\max}(K)) + \\ &+ \;\; (\mathcal{E}_{\max}(K) - z)^{m+1} \ln(z - \mathcal{E}_{\max}(K)) \Phi_{11}(K,z) + \widetilde{\Phi}_2(K,z), \end{aligned}$$

if $d = 2m + 2$ and

$$\nu(K,z) = \frac{\pi \Phi_0(K)}{2} \frac{(\mathcal{E}_{\max}(K) - z)^m}{\sqrt{z - \mathcal{E}_{\max}(K)}} + (\mathcal{E}_{\max}(K) - z)^{m+1/2} \, \Phi_{12}(K,z) + \widetilde{\Phi}_2(K,z),$$

if $d = 2m + 1$, where $z > \mathcal{E}_{\max}(K)$ and

$$\Phi_0(K) = \frac{C_0(K)}{\sqrt{\cos \dfrac{K_1}{2} \dots \cos \dfrac{K_d}{2}}}$$

and $\Phi_{11}(K,\cdot)$, $\widetilde{\Phi}_2(K,\cdot)$ are regular functions in some neighborhood $(\mathcal{E}_{\max}(K),\mathcal{E}_{\max}(K)+\xi)$ of $z=\mathcal{E}_{\max}(K)$. Since $\nu(K)=\widetilde{\Phi}_2(K,\mathcal{E}_{\max}(K))$, by the regularity of $\widetilde{\Phi}_2(K,z)$ we can rewrite $\widetilde{\Phi}_2(K,z)$ as following:

$$\widetilde{\Phi}_2(K,z)=\nu(K)+\Phi_2(K,z),$$

where $\Phi_2(K,\cdot)$ is also regular function in $(\mathcal{E}_{\max}(K),\infty)$ and $\Phi_2(K,\mathcal{E}_{\max}(K))=0$. □

Corollary 4.3. *The function $\nu(K,z)$ can be analytically continued to some punctured δ_1 - neighborhood $\dot{V}_{\delta_1}(\mathcal{E}_{\max}(K))$ of $\mathcal{E}_{\max}(K)$ as*

$$\nu^*(K,z) = \widetilde{\Phi}_{11}(K,z)\,\mathrm{Ln}(z-\mathcal{E}_{\max}(K))+\Phi_2(K,z), \quad \textit{if } d=2m+2$$

and

$$\nu^*(K,z) = \sum_{l=0}^{\infty} b_l(K)\ (z-\mathcal{E}_{\max}(K))^{(m+l)/2}+\Phi_2(K,z), \quad \textit{if } d=2m+1.$$

Proof. Note that the proposition of Lemma 4.2 is still hold if we change a real half neighborhood $(\mathcal{E}_{\max}(K),\mathcal{E}_{\max}(K)+\xi)$ to a punctured complex neighborhood $\dot{V}_{\delta_1}(\mathcal{E}_{\max}(K))$ of $\mathcal{E}_{\max}(K)$. Therefore the function $\nu(K,\cdot)$ may be analytically continued to $\dot{V}_{\delta_1}(\mathcal{E}_{\max}(K))$. □

Remark 4.4. *The proof of the Lemma yields that in case $d=1$ the function $\nu(K,z)$ has precise form:*

$$\nu(K,z)=\frac{\pi}{\sqrt{\cos\frac{K}{2}}}\frac{1}{\sqrt{z-\mathcal{E}_{\max}(K)}}-\frac{\pi}{8\sqrt{\cos^3\frac{K}{2}}}\sqrt{z-\mathcal{E}_{\max}(K)}+$$

$$+\frac{3\pi}{128\sqrt{\cos^5\frac{K}{2}}}\sqrt{(z-\mathcal{E}_{\max}(K))^3}-\frac{5\pi}{1024\sqrt{\cos^7\frac{K}{2}}}\sqrt{(z-\mathcal{E}_{\max}(K))^5}+\ldots$$

Proof of Theorem 3.2. The existence of unique eigenvalue $z(\mu,K)$ is a simple consequence of the Birman-Schwinger principle and the Fredholm theorem.

Indeed, for any fixed $K\in\Pi_0$ the function $\Delta_\mu(K,\cdot)$ is analytic and monotone increasing in $(\mathcal{E}_{\max}(K),\infty)$. Moreover for $\mu>\mu_0(K)$

$$\lim_{z\to+\infty}\Delta_\mu(K,z)=1 \text{ and } \lim_{z\to\mathcal{E}_{\max}(K)}\Delta_\mu(K,z)=1-\frac{\mu}{\mu_0(K)}<0. \tag{4.9}$$

Hence there is a unique number $z(\mu,K)\in(\mathcal{E}_{\max}(K),\infty)$ such that

$$\Delta_\mu(K,z(\mu,K))=0.$$

According to Lemma 4.1, $z(\mu,K)$ is an eigenvalue of the operator $H_\mu(K)$.

Let $\mu(K,z)=[\nu(K,z)]^{-1}$. The function $\mu(K,\cdot)$ is monotone and regular on $(\mathcal{E}_{\max}(K)),+\infty)$. Since $d\geq 3$, there exists finite limit

$$\lim_{z\to\mathcal{E}_{\max}(K)+0}\mu(K,z)=\mu(K,\mathcal{E}_{\max}(K)),$$

hence, we can redefine this function at the point $\mathcal{E}_{\max}(K) \in \mathbb{R}$ as $\mu(K, \mathcal{E}_{\max}(K)) = \mu_0(K)$. By the theorem on the existence of inverse monotone function, there exists such inverse function $z(\cdot, K) : [\mu_0(K), +\infty) \to [\mathcal{E}_{\max}(K)), +\infty)$ that the relation

$$\Delta_\mu(K, z(\mu, K)) = 0$$

holds for all $\mu \in [\mu_0(K), +\infty)$. By the regularity of the function $\mu(K, \cdot)$ in $(\mathcal{E}_{\max}(K)), +\infty)$ it follows that $z(\cdot, K)$ is also regular in $(\mu_0(K), +\infty)$. By the continuity and monotonicity of the functions $\mu(K, \cdot)$ and $z(\cdot, K)$ at $z = \mathcal{E}_{\max}(K)$ and $\mu = \mu_0(K)$, respectively, we get $\mu \to \mu_0(K)$ if and only if $z \to \mathcal{E}_{\max}(K)$.

The proves of parts (i)-(iv) of Theorem 3.2 are based on Lemma 4.2.

Taking into account

$$\frac{1}{\mu(K, z)} - \frac{1}{\mu_0(K)} = \nu(K, z) - \nu(K)$$

we get

$$-\frac{\mu(K, z) - \mu_0(K)}{\mu_0(K)(\mu(K, z) - \mu_0(K)) + (\mu_0(K))^2} = \nu(K, z) - \nu(K). \tag{4.10}$$

(i). Let $d = 3$. By Lemma 4.2 we can rewrite (4.10) as following:

$$-\frac{\lambda}{\mu_0\lambda + \mu_0^2} = f(\alpha, K), \tag{4.11}$$

where $\lambda = \mu(K, z) - \mu_0(K)$, $\alpha = (z - \mathcal{E}_{\max}(K))^{1/2}$, $\mu_0 = \mu_0(K)$ and $f(\alpha, K)$ is regular function in some neighborhood of $\alpha = 0$ with

$$f(0, K) = 0, \quad \frac{\partial f}{\partial \alpha}(0, K) = -\frac{\pi \Phi_0(K)}{2} < 0.$$

The proof of part (i) of Theorem 3.2 follows from Theorem A.1 (see Appendix A).

(ii). Let $d = 4$. By Lemma 4.2, we can rewrite (4.10) as following:

$$\lambda = -(\mu_0\lambda + \mu_0^2)(f(\alpha, K) + g(\alpha, K) \ln \alpha)$$

where $\lambda = \mu(K, z) - \mu_0(K)$, $\alpha = z - \mathcal{E}_{\max}(K)$, $\mu_0 = \mu_0(K)$ and $f(\cdot, K)$, $g(\cdot, K)$ are regular functions in some neighborhood of $\alpha = 0$ with

$$f(0, K) = g(0, K) = 0, \quad g'(0, K) = \frac{\Phi_0(K)}{2} > 0.$$

The proof of this part of the theorem follows from Theorem A.2 (see Appendix A).

(iii). Let $d \geq 5$ and odd. Since $K \in \Pi_0$ then $\nu(K,\cdot)$ is differentiable in $[\mathcal{E}_{\max}(K), +\infty)$ and

$$\frac{\partial \nu}{\partial z}(K, \mathcal{E}_{\max}(K)) = -\int\limits_{\mathbb{T}^d} \frac{dq}{(\mathcal{E}_{\max}(K) - \mathcal{E}_K(q))^2} < 0$$

for all $K \in \Pi_0$. Therefore, by Lemma 4.2 we get

$$\nu(K,z) = \nu(K) + \frac{\partial \nu}{\partial z}(K, \mathcal{E}_{\max}(K))(z - \mathcal{E}_{\max}(K)) + \sum_{n \geq 4} (z - \mathcal{E}_{\max}(K))^{n/2}.$$

We can rewrite (4.10) as following:

$$-\frac{\lambda}{\mu_0 \lambda + \mu_0^2} = f(\alpha, K),$$

where

$$\lambda = \mu(K,z) - \mu_0(K), \quad \alpha = (z - \mathcal{E}_{\max}(K))^{1/2}, \quad \mu_0 = \mu_0(K) \tag{4.12}$$

and $f(\cdot, K)$ is regular function in some neighborhood of $\alpha = 0$ with

$$f(0,K) = 0, \quad \frac{\partial f}{\partial \alpha}(0,K) = 0, \quad \frac{\partial^2 f}{\partial \alpha^2}(0,K) = -2\int\limits_{\mathbb{T}^d} \frac{dq}{(\mathcal{E}_{\max}(K) - \mathcal{E}_K(q))^2} < 0.$$

The proof of part (iii) of the theorem follows from Theorem A.3 (see Appendix A).

(iv). Let $d \geq 6$ and even (if $d = 2m+2$ then $m \geq 2$). Note that in this case $\nu(K,z)$ is represented as

$$\nu(K,z) = \nu(K) + \sum_{n \geq 1} A_n(K)(z - \mathcal{E}_{\max}(K))^n -$$

$$-\frac{\Phi_0(K)}{2} (\mathcal{E}_{\max}(K) - z)^m \ln(z - \mathcal{E}_{\max}(K)) \sum_{n \geq 1} B_n(K)(z - \mathcal{E}_{\max}(K))^n.$$

Then equation (4.10) takes the form

$$-\frac{\lambda}{\mu_0 \lambda + \mu_0^2} = f(\alpha, K) + g(\alpha, K) \, \ln \alpha,$$

where

$$\lambda = \mu(K,z) - \mu_0(K), \quad \alpha = (z - \mathcal{E}_{\max}(K))^{1/2}, \quad \mu_0 = \mu_0(K) \tag{4.13}$$

and functions $f(\cdot, K)$, $g(\cdot, K)$ are regular functions in the some neighborhood of $\alpha = 0$ with

$$f(0,K) = \frac{\partial f}{\partial \alpha}(0,K) = 0, \quad A_1(K) = \frac{\partial^2 f}{\partial \alpha^2}(0,K) = -2\int\limits_{\mathbb{T}^d} \frac{dq}{(\mathcal{E}_{\max}(K) - \mathcal{E}_K(q))^2} < 0,$$

$$g(0,K) = \frac{\partial g}{\partial \alpha}(0,K) = \frac{\partial^2 g}{\partial \alpha^2}(0,K) = 0.$$

The proof of part (iv) of the theorem follows from Theorem A.4 (see Appendix A). □

Proof of Theorem 3.3. Note that for any $K \in \Pi_0 \setminus \{0\}$ the inequality $\mu_0(K) < \mu_0(0)$ holds. Therefore the first part of Theorem 3.2 implies that $H_{\mu_0(0)}(K)$ has a unique eigenvalue $z(\mu_0(0), K)$ above the essential spectrum $\sigma_{ess}(H_{\mu_0(0)}(K))$.

Since the function $\nu(K, z)$ is real-analytic in $\mathbb{T}^d \times \mathbb{R} \setminus [\mathcal{E}_{\min}(K), \mathcal{E}_{\max}(K)]$ and the function $z(\mu_0(K), K)$ is solution of the equation $\Delta_{\mu_0(K)}(K, z) = 0$, the implicit function theorem implies that $z(\mu_0(0), K)$ is real-analytic function in $\Pi_0 \subset \mathbb{T}^d$. Observe that for the eigenvalue of $H_{\mu_0(0)}(K)$ the relation $z(\mu_0(0), K) \to \mathcal{E}_{\max}(0)$ holds as $K \to 0$.

According to the proof of Lemma 4.2, the functions $\nu(K)$ and $\Phi_0(K)$ can be written as

$$\nu(K) = \nu(0) + a_2 \sum_{l=1}^{d} K_l^2 + \sum_{l,m=1}^{d} a_{4,l,m} K_l^2 K_m^2 + O(|K|^4), \tag{4.14}$$

with $a_2 = \dfrac{\partial^2 \nu(0)}{\partial K_1^2}$ and

$$\Phi_0(K) = \frac{c}{\sqrt{\cos\dfrac{K_1}{2} \ldots \cos\dfrac{K_d}{2}}} = c_{0,0} + O(|K|^2),\ K \to 0, \quad c_{0,0} \neq 0. \tag{4.15}$$

(i) Let $z_0(K) = z(\mu_0(0), K)$ is eigenvalue of $H_{\mu_0(0)}(K)$. Then it is clear that

$$\Delta_{\mu_0(0)}(K, z_0(K)) = 1 - \mu_0(0) \int\limits_{\mathbb{T}^d} \frac{dq}{z_0(K) - \mathcal{E}_K(q)} = 0. \tag{4.16}$$

By Lemma 4.2 the following relation

$$0 = 1 - \mu_0(0)\Big(\frac{\pi \Phi_0(K)}{2}(z_0(K) - \mathcal{E}_{\max}(K))^{1/2} +$$
$$\nu(K) + (z_0(K) - \mathcal{E}_{\max}(K))O(1)\Big),\ K \to 0$$

holds. Since $\nu(0) = (\mu_0(0))^{-1}$ we get

$$(z_0(K) - \mathcal{E}_{\max}(K))^{1/2}\left[\frac{\pi \Phi_0(K)}{2} + (z_0(K) - \mathcal{E}_{\max}(K))^{1/2}O(1)\right] = -(\nu(K) - \nu(0))$$

and hence (4.15) and (4.14) imply that

$$(z_0(K) - \mathcal{E}_{\max}(K))\Big[c_{0,0} + O(|K|^2) + (z_0(K) - \mathcal{E}_{\max}(K))^{1/2}O(1)\Big]^2 =$$
$$= |K|^4(a_2 + a_4|K|^2 + o(|K|^2))^2,\ K \to 0.$$

According to the regularity of $z_0(K)$ on Π_0 and $c_{0,0} \neq 0$ we get that

$$z_0(K) - \mathcal{E}_{\max}(K) = c_{0,0}^{-2} a_2^2 |K|^4 + o(|K|^4),\ \text{as } K \to 0.$$

Using $\mathcal{E}_{\max}(K)=\mathcal{E}_{\max}(0)-\frac{1}{4}|K|^2+O(|K|^4),\ K\to 0$, we establish

$$z(\mu_0(0),K)=\mathcal{E}_{\max}(0)-\frac{1}{4}|K|^2+O(|K|^4),\ K\to 0.$$

(ii) Taking into account (4.14), using Lemma 4.2 and the fact that $z_0(K)=z(\mu_0(0),K)$ is a solution of $\Delta_{\mu_0(0)}(K,z)=0$, we rewrite (4.16) as following:

$$(z_0(K)-\mathcal{E}_{\max}(K))\ln[z_0(K)-\mathcal{E}_{\max}(K)]+(z_0(K)-\mathcal{E}_{\max}(K))\,O(1)=$$
$$|K|^2(c+O(|K|^2)),\ K\to 0$$

where $c\neq 0$. Since

$$0=\lim_{K\to 0}\frac{z_0(K)-\mathcal{E}_{\max}(K)}{(z_0(K)-\mathcal{E}_{\max}(K))\ln(z_0(K)-\mathcal{E}_{\max}(K))+(z_0(K)-\mathcal{E}_{\max}(K))\,O(1)}=$$
$$=\lim_{K\to 0}\frac{z_0(K)-\mathcal{E}_{\max}(K)}{|K|^2(c+O(|K|^2))}$$

we obtain $z_0(K)-\mathcal{E}_{\max}(K)=o(|K|^2)$, as $K\to 0$. Consequently

$$z(\mu_0(0),K)-\mathcal{E}_{\max}(0)=-\frac{1}{4}|K|^2+o(|K|^2),\ K\to 0.$$

(iii) It is easy to get that

$$\frac{\partial\nu(K,\mathcal{E}_{\max}(K))}{\partial z}=\frac{\partial\nu(0,\mathcal{E}_{\max}(0))}{\partial z}+O(|K|),\quad K\to 0. \tag{4.17}$$

Using Taylor expansion of $\nu(K,z)$, we obtain

$$\nu(K,z)=\nu(K)+\frac{\partial\nu(K,\mathcal{E}_{\max}(K))}{\partial z}(z-\mathcal{E}_{\max}(K))+(z-\mathcal{E}_{\max}(K))o(1),$$

as $z\to\mathcal{E}_{\max}(K)$ and taking into account that $z_0(K)=z(\mu_0(0),K)$ is solution of the equation $\Delta_{\mu_0(0)}(K,z)=0$, we get

$$-(\nu(K)-\nu(0))=\frac{\partial\nu(K,\mathcal{E}_{\max}(K))}{\partial z}(z_0(K)-\mathcal{E}_{\max}(K))+(z_0(K)-\mathcal{E}_{\max}(K))o(1),$$

as $K\to 0$, which means that

$$z_0(K)-\mathcal{E}_{\max}(K)=\frac{a_2|K|^2+o(|K|^2)}{-\frac{\partial\nu}{\partial z}(0,\mathcal{E}_{\max}(0))+O(|K|)}=\frac{a_2}{-\frac{\partial\nu}{\partial z}(0,\mathcal{E}_{\max}(0))}|K|^2+o(|K|^2).$$

Consequently

$$z(\mu_0(0),K)-\mathcal{E}_{\max}(0)=\left(\frac{a_2}{-\frac{\partial\nu}{\partial z}(0,\mathcal{E}_{\max}(0))}-\frac{1}{4}\right)|K|^2+o(|K|^2),\ K\to 0,$$

where $a_2=\dfrac{\partial^2\nu(0)}{\partial K_1^2}$. $\square$

APPENDIX A. CONSEQUENCES OF THE IMPLICIT FUNCTION THEOREM

In this appendix we want to prove results which are used in Theorem 3.2. Such kinds of lemmas are considered in [1]. By making suitable substitutions, we will reduce the proof to the implicit function theorem in several variables.

Theorem A.1. *Let $f(\alpha)$ be analytic function in a neighborhood of $\alpha = 0$. Suppose that*

$$f(0) = 0, \quad f'(0) < 0.$$

Then for λ sufficiently small there is a unique positive $\alpha(\lambda)$ satisfying

$$\lambda = -(\mu_0\lambda + \mu_0^2)f(\alpha). \tag{A.1}$$

Moreover, for λ sufficiently small, α has a convergent expansion

$$\alpha = \sum_{n\geq 1} c_n\lambda^n, \tag{A.2}$$

with $c_1 = -[\mu_0^2 f'(0)]^{-1}$.

Proof. We can write

$$f(\alpha) = \sum_{n\geq 1} a_n\alpha^n, \quad a_1 = f'(0) < 0.$$

In (A.9) try substitution

$$\alpha = \lambda(c+\xi), \quad c = -(a_1\,\mu_0^2)^{-1}.$$

Then in the region where $|\alpha|$ sufficiently small this is equivalent to

$$a_1\mu_0^2\xi = -(c+\xi)\left\{a_1\mu_0\lambda + (\mu_0\lambda+\mu_0^2)\sum_{n\geq 2} a_n\lambda^{n-1}(c+\xi)^{n-1}\right\} \tag{A.3}$$

Equation (A.3) can be written in the form

$$F(\xi,\lambda) = 0$$

where (i) $\xi = 0$, $\lambda = 0$ is a solution; (ii) F is analytic for $|\lambda|$, $|\xi|$ small; (iii) $\partial F/\partial\xi(0,0) = -a_1\mu_0^2 \neq 0$. Thus by the implicit function theorem, (A.3) has a unique solution for ξ, λ small given by a convergent expansion

$$\xi = \sum_{n\geq 0} b_n\lambda^n.$$

Consequently,

$$\alpha = \lambda(c+\xi) = \sum_{n\geq 1} c_n\lambda^n,$$

where $c_1 = c = -[\mu_0^2\, f'(0)]^{-1}$. $\square$

Theorem A.2. *Let $f(\alpha)$, $g(\alpha)$ be analytic functions in a neighborhood of $\alpha = 0$. Suppose that*

$$f(0) = g(0) = 0, \quad g'(0) > 0.$$

Then for λ sufficiently small and positive there is a unique positive $\alpha(\lambda)$ satisfying

$$\lambda = -(\mu_0\lambda + \mu_0^2)\,(f(\alpha) + g(\alpha)\ln\alpha)\,. \tag{A.4}$$

Moreover, for λ sufficiently small, α has a convergent expansion

$$\alpha(\lambda) = \sum_{n\geq 1,\, m,k,l\geq 0} c(n,m,k,l)\;\sigma^n\tau^m\omega^k\lambda^l \tag{A.5}$$

with

$$\sigma = \frac{\lambda}{-\ln\lambda}, \quad \tau = \frac{1}{-\ln\lambda}, \quad \omega = \frac{\ln\ln\lambda^{-1}}{-\ln\lambda} \tag{A.6}$$

and $c(1,0,0,0) = (\mu_0^2\;\, g'(0))^{-1}$.

Proof. We write

$$f(\alpha) = \sum_{n\geq 1} a_n\alpha^n,$$

$$g(\alpha) = \sum_{n\geq 1} b_n\alpha^n, \quad b_1 = g'(0) > 0.$$

Try the substitution

$$\alpha = \sigma\,(c + \xi), \quad c = (b_1\,\mu_0^2)^{-1}. \tag{A.7}$$

Then, it is easy to find that in the region where $|\alpha|$ is small, (A.12) is equivalent to

$$b_1\,\mu_0^2\,\xi = (c+\xi)\Bigg\{-\mu_0\lambda b_1 + (\mu_0\lambda + \mu_0^2)\Big[\tau\sum_{n\geq 0} a_{n+1}\sigma^n(c+\xi)^n - \tag{A.8}$$

$$-\sum_{n\geq 1} b_{n+1}\sigma^n(c+\xi)^n - \omega\sum_{n\geq 0} b_{n+1}\sigma^n(c+\xi)^n - \tau\ln(c+\xi)\sum_{n\geq 0} b_{n+1}\sigma^n(c+\xi)^n\Big]\Bigg\}.$$

Equation (A.8) can be written in the form

$$F(\xi,\sigma,\tau,\omega,\lambda) = 0,$$

where (i) $\xi = 0$, $\sigma = 0$, $\tau = 0$, $\omega = 0$, $\lambda = 0$ is a solution; (ii) F is analytic for $|\xi|$, $|\sigma|$, $|\tau|$, $|\omega|$, $|\lambda|$ small since $\ln(c+\xi)$ is analytic at $\xi = 0$; (iii) $\partial F/\partial\xi(0,0,0,0,0) = b_1\mu_0^2 \neq 0$. Thus by the implicit function theorem, (A.8) has a unique solution for ξ, σ, τ, ω, λ small given by a convergent expansion

$$\xi = \sum_{n,m,k,l\geq 0} d(n,m,k,l)\,\sigma^n\,\tau^m\,\omega^k\,\lambda^l.$$

Given (A.15) and (A.14), this yields (A.13). $\square$

Theorem A.3. *Let $f(\alpha)$ be analytic function in a neighborhood of $\alpha = 0$. Suppose that*

$$f(0) = f'(0) = 0, \quad f''(0) < 0.$$

Then for λ sufficiently small there is a unique positive $\alpha(\lambda)$ satisfying

$$\lambda = -(\mu_0\lambda + \mu_0^2) f(\alpha). \tag{A.9}$$

Moreover, for λ sufficiently small, α has a convergent expansion

$$\alpha = \left(\sum_{n\geq 1} c_n \lambda^{n/2} \right)^2, \tag{A.10}$$

with $c_1 = [-1/2\mu_0^2\, f''(0)]^{-1/2}$.

Proof. We can write

$$f(\alpha) = \sum_{n\geq 2} a_n \alpha^n, \quad a_2 = 1/2 f''(0) < 0.$$

In (A.9) try substitution

$$\alpha = \sigma(c+\xi), \quad c = (-a_2\, \mu_0^2)^{-1/2},$$

where $\sigma = \lambda^{1/2}$. Then in the region where $|\alpha|$ sufficiently small this is equivalent to

$$2\mu_0\sqrt{-a_2}\,\xi - a_2\mu_0^2\,\xi^2 = (c+\xi)^2 \left\{ \mu_0 a_2 \sigma^2 - (\mu_0\sigma^2 + \mu_0^2) \sum_{n\geq 3} a_n \sigma^{n-2} (c+\xi)^{n-2} \right\} \tag{A.11}$$

Equation (A.11) can be written in the form

$$F(\xi, \sigma) = 0$$

where (i) $\xi = 0$, $\sigma = 0$ is a solution; (ii) F is analytic for $|\sigma|$, $|\xi|$ small; (iii) $\partial F/\partial\xi(0,0) = 2\mu_0\sqrt{-a_2} \neq 0$. Thus by the implicit function theorem, (A.11) has a unique solution for ξ, σ small given by a convergent expansion

$$\xi = \sum_{n\geq 0} b_n \sigma^n.$$

Consequently,

$$\alpha = \sigma(c+\xi) = \sum_{n\geq 1} c_n \sigma^n,$$

where $c_1 = c = [-1/2\mu_0^2\, f''(0)]^{-1/2}$. □

Theorem A.4. *Let $f(\alpha)$, $g(\alpha)$ be analytic functions in a neighborhood of $\alpha = 0$. Suppose that*

$$f(0) = f'(0) = g(0) = g'(0) = g''(0) = 0, \quad f''(0) < 0, \quad .$$

Then for λ sufficiently small and positive there is a unique positive $\alpha(\lambda)$ satisfying

$$\lambda = -(\mu_0\lambda + \mu_0^2)\left(f(\alpha) + g(\alpha)\ln\alpha\right). \tag{A.12}$$

Moreover, for λ sufficiently small, α has a convergent expansion

$$\alpha(\lambda) = \sum_{n\geq 1,\, k\geq 0} c(n,k)\ \sigma^n \tau^k \tag{A.13}$$

with

$$\sigma = \lambda^{1/2}, \quad \tau = \sigma \ln \sigma \tag{A.14}$$

and $c(1,0) = (-1/2\mu_0^2\ \ f''(0))^{-1/2}$.

Proof. We write

$$f(\alpha) = \sum_{n\geq 2} a_n \alpha^n, \quad a_2 = 1/2 f''(0) < 0, \qquad g(\alpha) = \sum_{n\geq 3} b_n \alpha^n.$$

Try the substitution

$$\lambda = \sigma^2, \quad \alpha = \sigma\,(c+\xi), \quad c = (-a_2\,\mu_0^2)^{-1/2}. \tag{A.15}$$

Then

(A.16)

$$2\mu_0\sqrt{-a_2}\,\xi - \mu_0^2 a_2\,\xi^2 = (c+\xi)^2 \Big\{ - \mu_0\sigma^2 a_2 + (\mu_0\sigma^2 + \mu_0^2)\Big[\sum_{n\geq 3} a_n \sigma^{n-2}(c+\xi)^{n-2} +$$

$$+\tau \sum_{n\geq 3} b_n \sigma^{n-3}(c+\xi)^{n-2} + \ln(c+\xi)\sum_{n\geq 3} b_n\sigma^{n-2}(c+\xi)^{n-2}\Big]\Big\},$$

where $\tau = \sigma \ln \sigma$.

Equation (A.16) can be written in the form

$$F(\xi,\sigma,\tau) = 0,$$

where (i) $\xi = 0, \sigma = 0, \tau = 0$ is a solution; (ii) F is analytic for $|\xi|$, $|\sigma|$, $|\tau|$ small since $\ln(c+\xi)$ is analytic at $\xi = 0$; (iii) $\partial F/\partial\xi(0,0,0) = 2\mu_0\sqrt{-a_2} \neq 0$. Thus by the implicit function theorem, (A.8) has a unique solution for σ, τ small given by a convergent expansion

$$\xi = \sum_{n,k\geq 0} d(n,k)\,\sigma^n\,\tau^k.$$

Given (A.15) and (A.14), this yields (A.13). □

APPENDIX B.

In this section we give two lemmas which are used in the proof of Lemma 4.2.

Lemma B.1. *The function*

$$I_s(\theta) = \int_0^{\gamma} \frac{r^s}{r^2-\theta}, \quad \theta < 0,\ s = 0,1,2,\ldots$$

is represented as

$$I_s(\theta) = \begin{cases} -1/2\,\theta^m\, \ln(-\theta) + \hat{I}_s(\theta), & s\text{=}2m\text{+}1 \\ \dfrac{\pi}{2\sqrt{-\theta}}\,\theta^m + \tilde{I}_s(\theta), & s\text{=}2m \end{cases} \tag{B.1}$$

where $\hat{I}_s(\theta)$ and $\tilde{I}_s(\theta)$ are regular functions in the some neighborhood of zero.

Proof. We will prove for $s = 2m + 1$. For the case $s = 2m$ it can be proved analogously. Integrating the identity

$$\frac{r^{2m+1}}{r^2 - \theta} = r\left(r^{2(m-1)} + \theta r^{2(m-2)} + \ldots + \theta^{m-1}\right) + \frac{r\theta^m}{r^2 - \theta}$$

from 0 to γ, we get

$$\int\limits_0^\gamma \frac{r^{2m+1}}{r^2 - \theta} = \frac{\gamma^{2m}}{2m} + \theta\frac{\gamma^{2(m-1)}}{2(m-1)} + \ldots + \theta^{m-1}\frac{\gamma^2}{2} + \frac{1}{2}\,\theta^m\, \ln\frac{\gamma^2 - \theta}{-\theta}$$

Therefore

$$\int\limits_0^\gamma \frac{r^{2m+1}}{r^2 - \theta} = -\frac{1}{2}\,\theta^m\, \ln(-\theta) + \hat{p}_{m-1}(\theta) + f_1(\theta),$$

Denote by

$$\hat{I}_m(\theta) = \hat{p}_{m-1}(\theta) + f_1(\theta),$$

where $\hat{p}_{m-1}(\theta)$ is a polynomial with degree $m - 1$, $f_1(\theta)$ is a regular function in some neighborhood of zero. $\square$

Lemma B.2.

$$a_n = \int\limits_{-\pi/2}^{\pi/2} \cos^n t dt = \begin{cases} \pi\dfrac{(2m-1)!!}{(2m)!!}, & n = 2m \\ 2\dfrac{(2m)!!}{(2m+1)!!}, & n = 2m+1 \end{cases} \tag{B.2}$$

Proof. Integrating by parts we easily receive the relation $a_n = \dfrac{n-1}{n} a_{n-2}$, where $n = 2, 3, 4, \ldots$. Using $a_0 = \pi$, $a_1 = 2$ we get (B.2). $\square$

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